Lecture 17 : Higher Derived Images

Note Title 12/5/2019

$f: X \longrightarrow Y$ continuous map between topological spaces

$f_*: Sh(X) \longrightarrow Sh(Y)$ left exact

$\rightsquigarrow$ $R^i f_*$ the associate right derived functor.

Proposition 1: $\forall\, i \geq 0$, $\mathcal{F} \in Sh(X)$

$R^i f_* \mathcal{F}$ is $\boxed{\text{the sheafification of the presheaf } V \longmapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})}$

pf:

$\mathcal{H}^i(X, \mathcal{F}) : Sh(X) \longrightarrow Sh(Y)$

$\mathcal{H}^0(X, \mathcal{F}) = f_* \mathcal{F}$

- If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact on X

  then $0 \to \mathcal{F}'|_V \to \mathcal{F}|_V \to \mathcal{F}''|_V \to 0$ exact on $f^{-1}(V) \underset{\text{open}}{\subseteq} X$

$\{H^i(f^{-1}(V), -)\}$ $\underline{\delta\text{-functor}}$ $\Rightarrow$ $\{\mathcal{H}^i(X, -)\}$ δ-functor

$\{T^i\}$ s.t. • $0 \to A \to B \to C \to 0$ exact

$\Rightarrow \cdots \to T^i(A) \to T^i(B) \to T^i(C) \to T^{i+1}(A) \to \cdots$

• natural transformation

Theorem 1 (Grothendieck) $\{T^i\}$ δ-functor from $\mathcal{A}$ to $\mathcal{B}$

s.t. $\forall A \in \mathcal{A}$, $\exists A \hookrightarrow \tilde{A}$ w/ $T^i(\tilde{A}) = 0$, $i > 0$

Then $T^i \cong R^i T^0$

Lemma 1: If $\mathcal{J}$ is an injective $\mathcal{O}_X$-module,

then $\mathcal{J}|_V$ is injective $\mathcal{O}_V$-module, for $V \overset{j}{\subseteq} X$ open

pf:

$$\begin{tikzcd} \mathcal{F} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathcal{G} \arrow[dl, dashed] \\ \mathcal{J}|_V & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} j_!\mathcal{F} \arrow[r, hook] \arrow[d] & j_!\mathcal{G} \arrow[dd, "\exists!"] \\ j_!(\mathcal{J}|_V) \arrow[dr] & \\ & \mathcal{J} \end{tikzcd}$$

$\mathcal{O}_V$-modules

extension by zero

$\because$ $\mathcal{J}$ injective $\mathcal{O}_X$-module

$\mathcal{O}_X$-modules

restrict to V

$\mathcal{G} \to \mathcal{J}|_V$

Proposition 2: $f: X \longrightarrow Y \cong \operatorname{Spec} A$, $\mathcal{F}$: quasi-coherent on X

(X Notherian scheme)

then $R^i f_* \mathcal{F} \cong \widetilde{H^i(X, \mathcal{F})}$

pf: • $T^i: \mathcal{F} \in \mathfrak{Qcoh}(X) \longrightarrow \widetilde{H^i(X,\mathcal{F})} \in \mathfrak{Qcoh}(Y)$ δ-functor

$\mathcal{F} \longmapsto H^i(X,\mathcal{F})$ δ-functor & $\sim$ exact

• $T^0(\mathcal{F}) = \widetilde{H^0(X,\mathcal{F})} = \widetilde{H^0(Y, f_*\mathcal{F})} = f_*\mathcal{F}$, $\forall \mathcal{F}$

$f_*\mathcal{F}$ quasi-coherent on Y

- Every quasi-coherent sheaf can be embedded into a flasque quasi-coherent sheaf.

$$\xrightarrow[\text{Theorem 1}]{} \quad T^i \cong R^i f_*$$

Corollary: $f: \underset{\text{Noetherian scheme}}{X} \longrightarrow Y$, $\mathcal{F}$: quasi-coherent on X

Then $R^i f_* \mathcal{F}$ is quasi-coherent on Y

check on each affine open subset & reduces to Proposition 2

Proposition 3. $f: X \longrightarrow Y$ separated morphism of Noetherian schemes

$\mathcal{F}$: quasi-coherent sheaf on X

$\{U_i\}$ affine open cover of X

then $R^i f_* \mathcal{F} = h^i\left(f_* \mathcal{C}^\bullet(\{U_i\}, \mathcal{F})\right)$

can be computed by Čech cohomology.

pf: • $\forall V \subseteq Y$ $f^{-1}(V) \cap U_i$ is affine

since $R^i f_* \mathcal{F}$ is quasi-coherent, may reduce to the case $Y = \operatorname{Spec} A$ affine.

$0 \to \mathcal{F} \to \mathcal{C}^\bullet(\{U_i\}, \mathcal{F})$ quasi-coherent, acyclic resolution

$\Longrightarrow \quad 0 \to \mathcal{F} \to f_* \mathcal{C}^\bullet(\{U_i\}, \mathcal{F}) = \widetilde{C^\bullet(\{U_i\}, \mathcal{F})}$ on $Y = \operatorname{Spec} A$

quasi-coherent

acyclic resolution

(Serre)
higher cohomology vanishes
of quasi-coherent sheaves
vanish on affine scheme.

Theorem 2. $f: X \to Y$ projective morphism, $\mathcal{O}_X(1)$ very ample between Noetherian schemes

$\mathcal{F}$: coherent sheaf on X

Then ① $f^* f_* (\mathcal{F}(n)) \twoheadrightarrow \mathcal{F}(n)$ for $n \gg 0$

② $R^i f_* \mathcal{F}$ is coherent sheaf on Y. $\forall i \geq 0$

③ $R^i f_* \mathcal{F}(n) = 0, \ \forall i > 0, \ n \gg 0.$

pf: • Projective morphism is preserved under base change.

+ Conditions in ① ② ③ can be checked locally.

$\rightsquigarrow$ reduce to the case $Y = \operatorname{Spec} A$ affine

① $\mathcal{F}(n)$ is generated by global sections, $n \gg 0$

② Proposition 2: $R^i f_* \mathcal{F} = \widetilde{H^i(X, \mathcal{F})}$ coherent

$\because H^i(X, \mathcal{F})$ finitely generated A-module

③ $R^i f_*(\mathcal{F}(n)) = \widetilde{H^i(X, \mathcal{F}(n))} = 0, \quad n \gg 0$